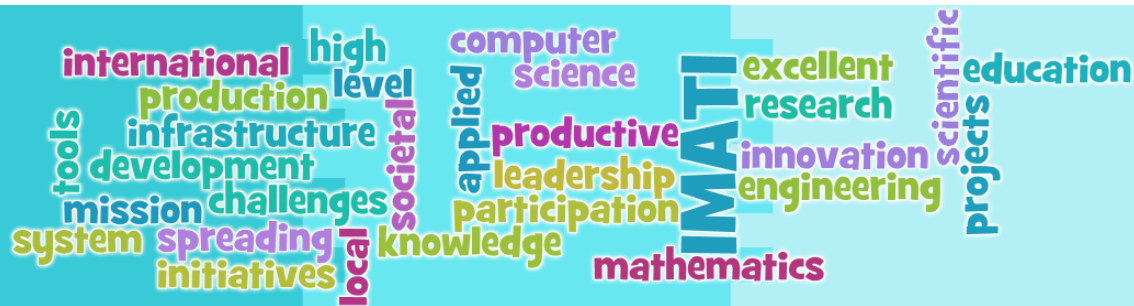


Extracting Geometrical Features From Data

Topological Data Analysis

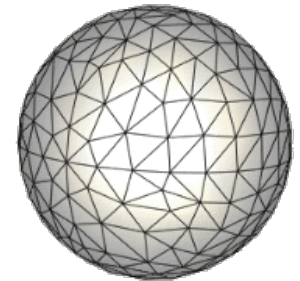
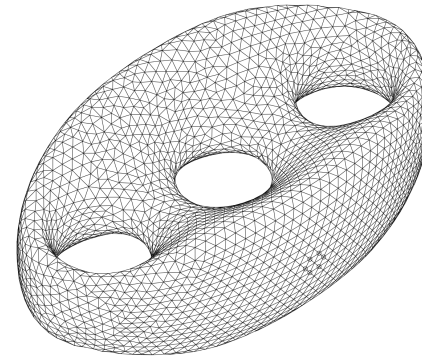
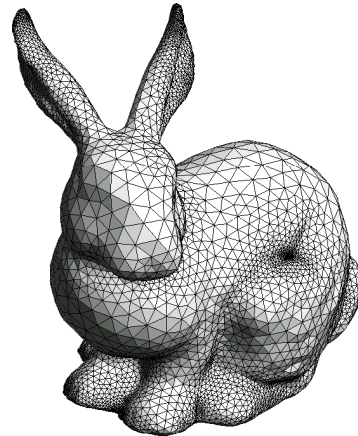
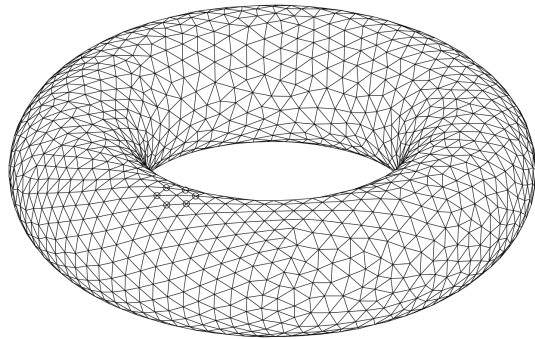
Ulderico Fugacci

CNR - IMATI



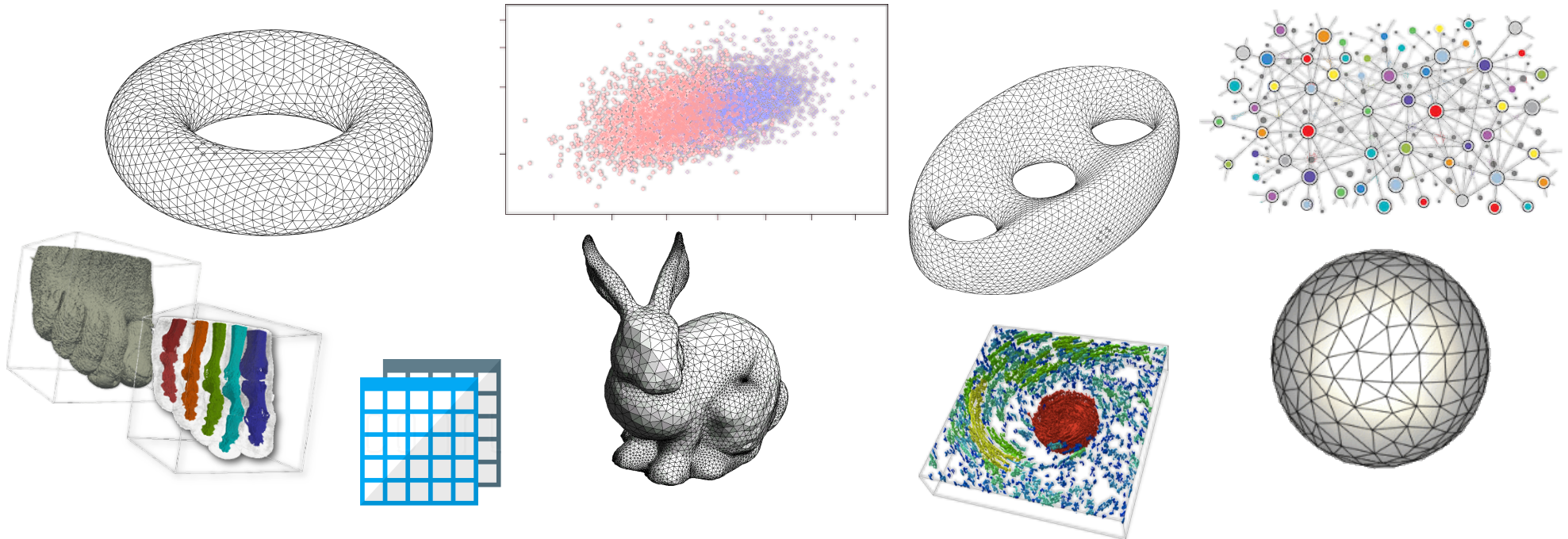
Topological Data Analysis

Topology describes, characterizes, and discriminates *shapes* by studying their properties that are preserved under *continuous deformations*, such as *stretching* and *bending*, but *not tearing* or *gluing*

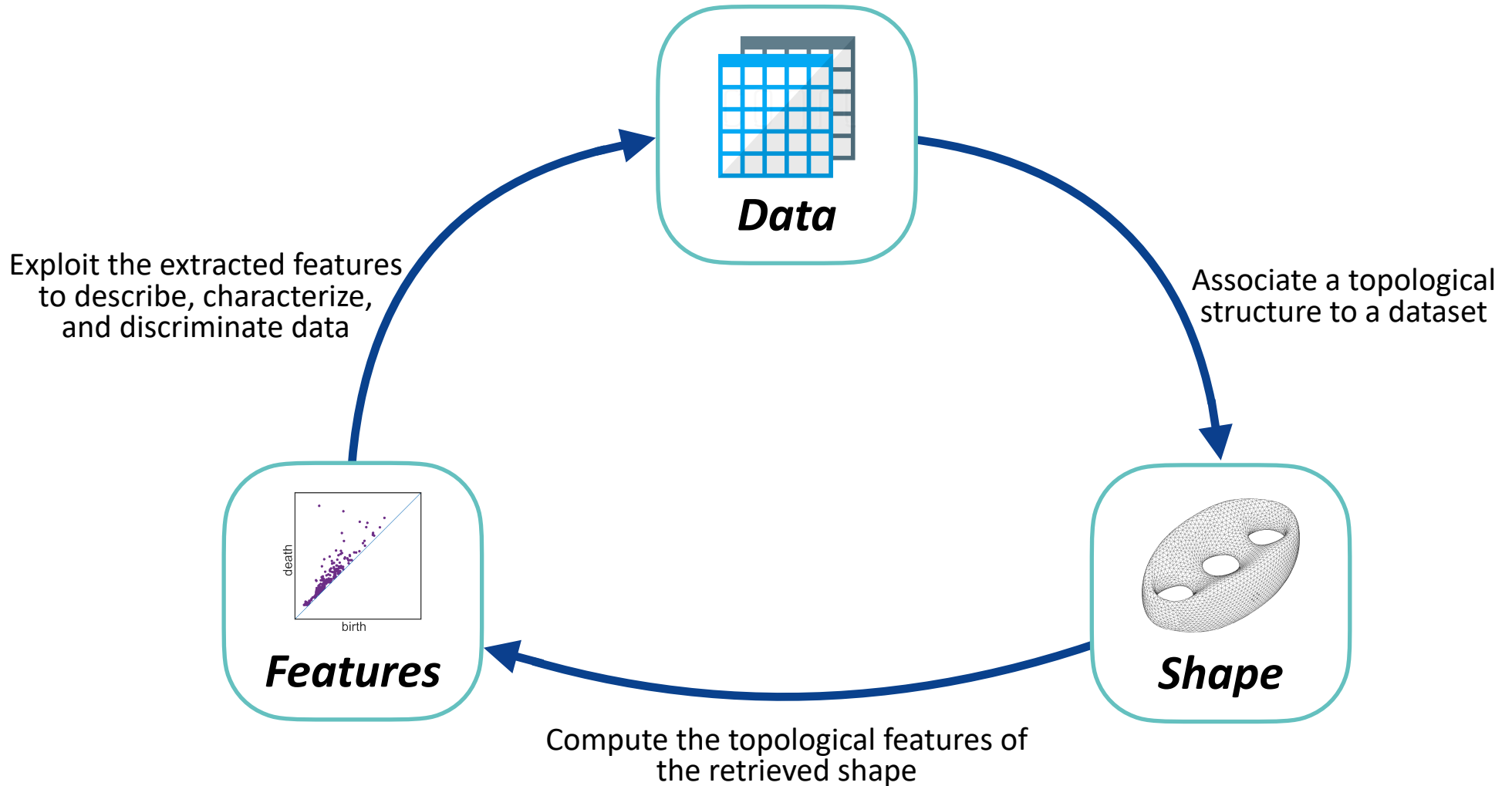


Topological Data Analysis

Assumption in TDA: *Any data* can be endowed with a *shape*.
So, any data can be studied in terms of its *topological features*



Topological Data Analysis



Topological Data Analysis

Outline:

The Notion of Shape

Simplicial Complexes

Simplicial Homology

From Data to Complexes

Persistent Homology

Visualizing Persistence

Persistence & Stability

Computing Persistence

The Notion of Shape

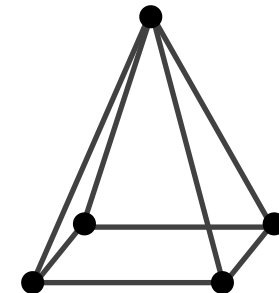
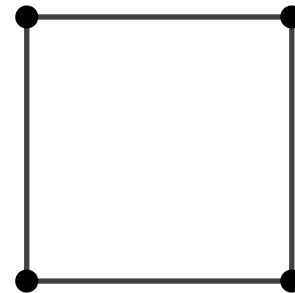
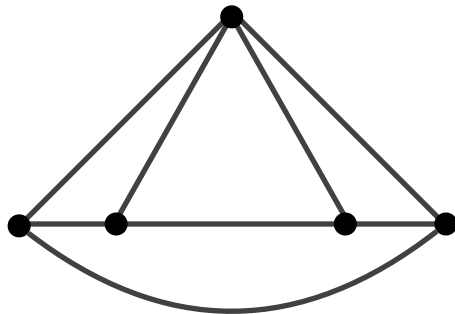
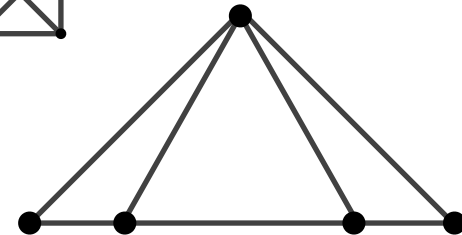
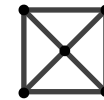
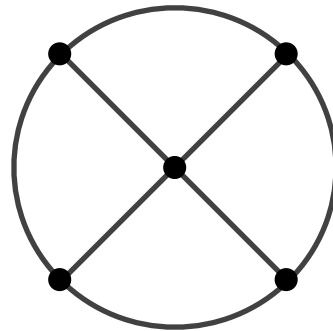
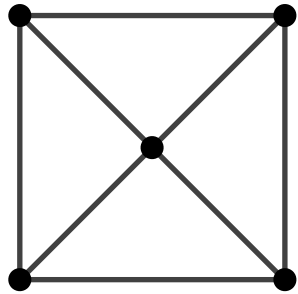
Geometry or Topology?

Which of these domains look **similar**?



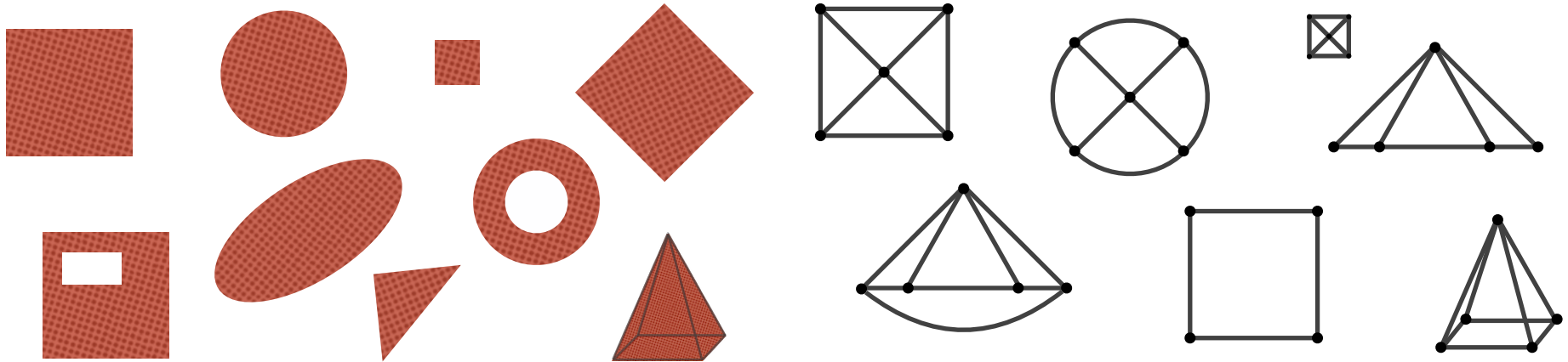
Geometry or Topology?

And what about these ones?



Geometry or Topology?

The answer depends on the *point of view* we adopt

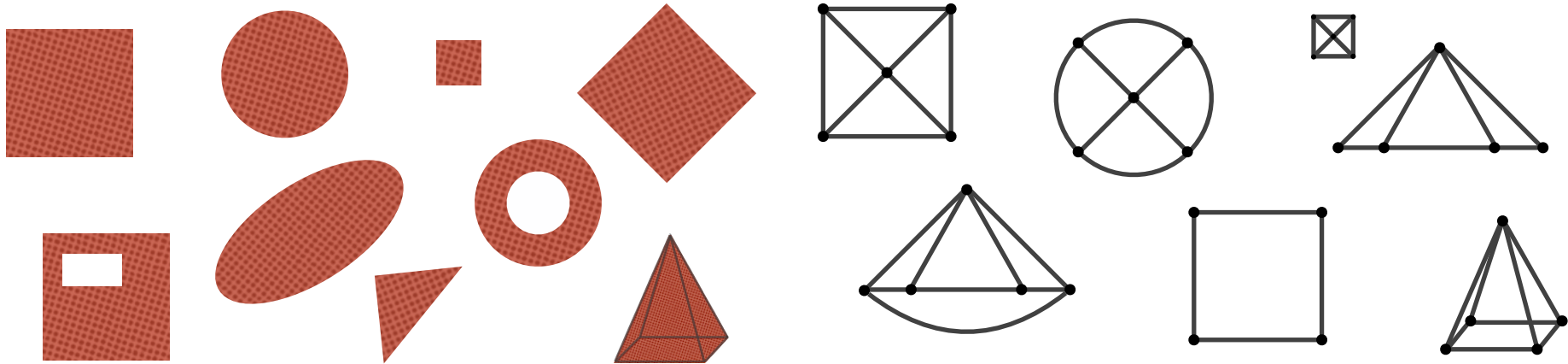


Geometry cares about those properties which **change** when an object is continuously **deformed**

E.g. length, area, volume, angles, curvature, ...

Geometry or Topology?

The answer depends on the *point of view* we adopt



Topology

~~Geo~~*metry* cares about those properties which *do not* change when an object is continuously **deformed**

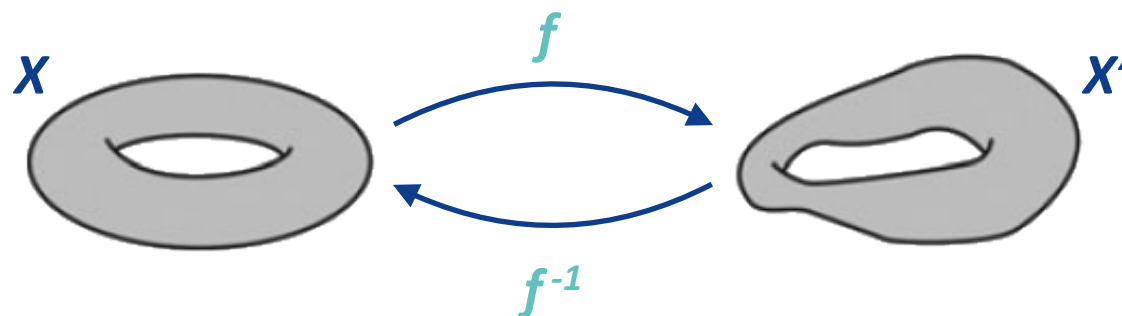
E.g. connectivity, orientation, manifoldness, ...

Homeomorphisms

Definition:

Given two topological spaces (X, T) and (X', T') , a function $f: X \rightarrow X'$ is called **homeomorphism** if:

- ♦ f is a **bijection**
- ♦ f is **continuous**
- ♦ f^{-1} is **continuous**

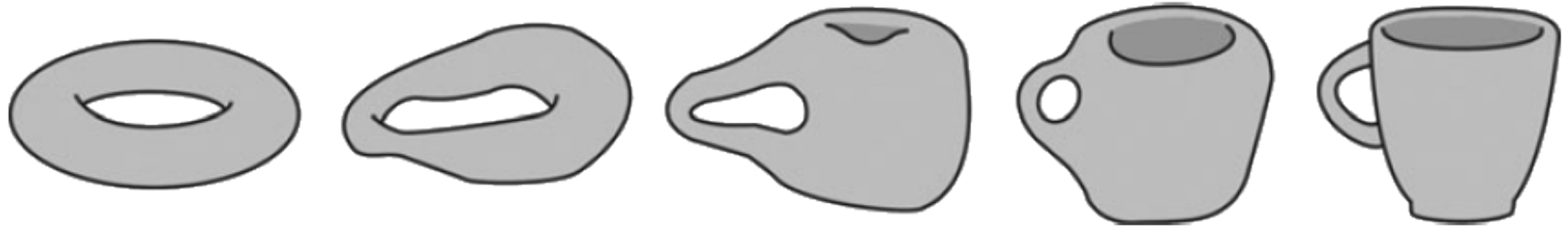


Two topological spaces (X, T) and (X', T') are **homeomorphic** and denoted $X \cong X'$ if there exists a homeomorphism $f: X \rightarrow X'$

Homeomorphisms induce an **equivalence relation** of topological spaces partitioning them into equivalence classes

Homeomorphisms

Intuitively:



*The notion of homeomorphism captures the idea of **continuous deformation***



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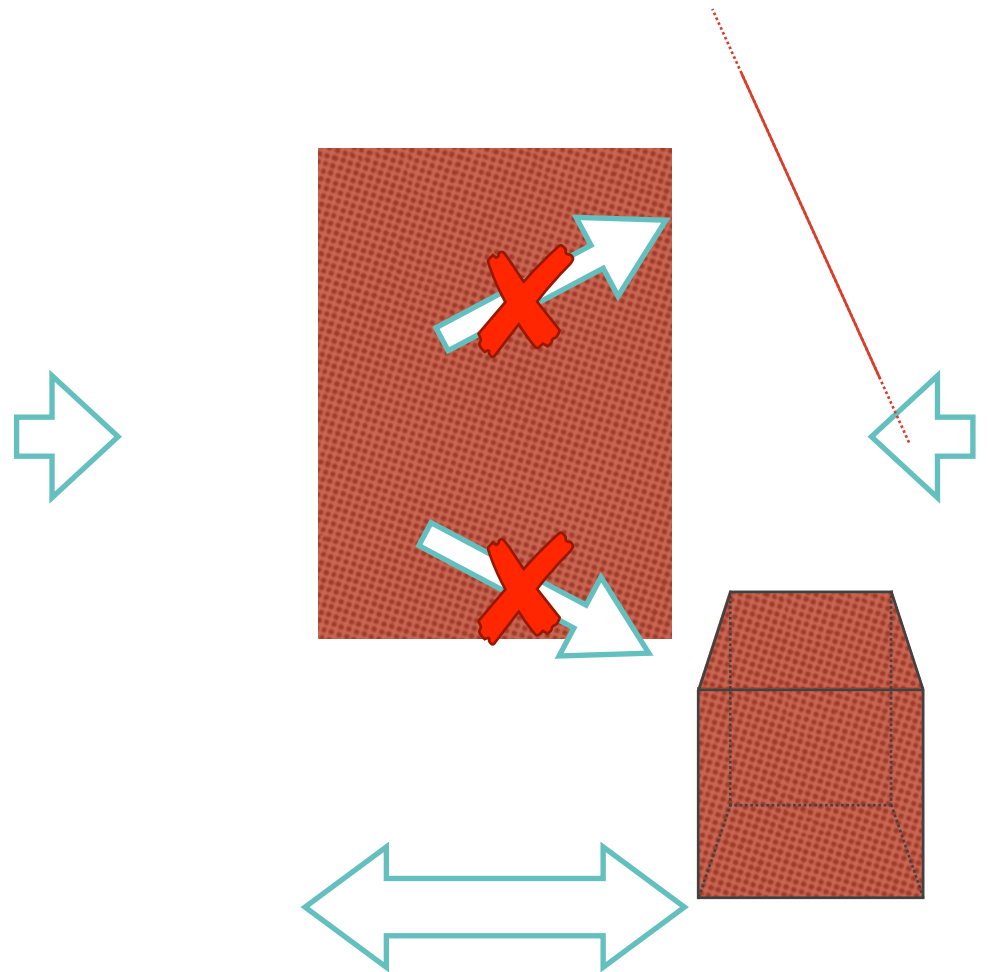
Homeomorphisms

Intuitively:

One can:

- ◆ Stretch
- ◆ Compress

But not too much!



Homeomorphisms

Intuitively:

Moreover:

- ◆ No Cut
- ◆ No Glue

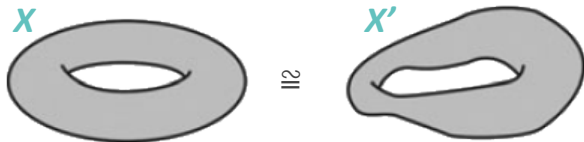


Topological Invariants

Definition:

I is a **topological invariant** if, given two topological spaces (X, T) and (X', T') ,

X is homeomorphic to X'



X and X' have the same
topological invariant

$$I(X) = I(X')$$

Some classical topological invariants:

- ◆ *Connectedness*
- ◆ *Compactness*
- ◆ *Manifoldness*
- ◆ *Orientability*
- ◆ *Euler characteristic*
- ◆ *Homology*
- ◆ *Homotopy*

Topological Invariants

Question:

Is there a “perfect” topological invariant I such that

$X \cong X'$ if and only if $I(X) = I(X')$?

Let us **simplify the question** and let focus on:

- ◆ Considering a specific topological invariant I (e.g. the **homology**)
- ◆ Completely characterizing just the **spheres** $S^n := \{x \in \mathbb{R}^n : |x| = 1\}$

The above question turns into the following:

If X and S^n have the same homology, then $X \cong S^n$?

NO

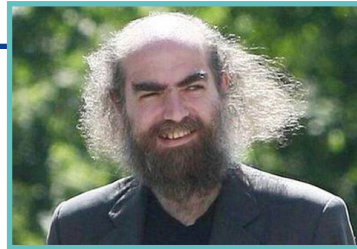
Topological Invariants

But:

Replacing homology with **homotopy**, the answer is positive!

Poincaré Conjecture (3rd Millennium Prize Problem):

If X is a closed n -manifold homotopy equivalent to S^n , then $X \cong S^n$



Proven by Grigori Perelman in 2003

So:

Why we will mainly focus on homology rather than homotopy?

Because, in practice, computing homotopy groups is nearly impossible!

Bibliography

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- ❖ G. Carlsson. *Topology and data*. Bulletin of the American Mathematical Society 46.2, pages 255-308, 2009.

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- ❖ A. Hatcher. *Algebraic topology*. Cambridge University Press, 2002.